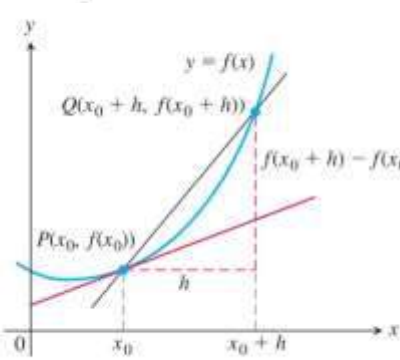
# Derivatives

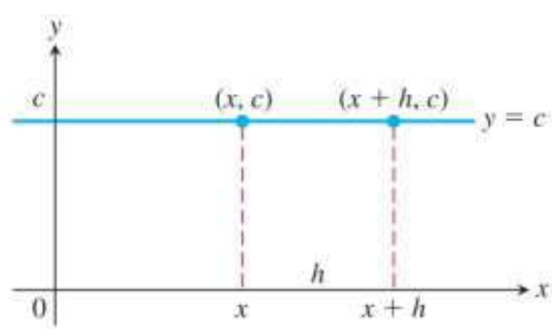
## Derivatives

**DEFINITIONS** The **slope of the curve** $y = f(x)$ at the point $P(x_0, f(x_0))$ is the number

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \quad \text{(provided the limit exists).}$$

The **tangent line** to the curve at $P$ is the line through $P$ with this slope.



# Derivatives
Derivatives

### Derivative of a Constant Function

If $f$ has the constant value $f(x) = c$, then

$$\frac{df}{dx} = \frac{d}{dx}(c) = 0.$$

### Derivative of a Positive Integer Power

If $n$ is a positive integer, then

$$\frac{d}{dx}x^n = nx^{n-1}.$$

### Derivative Constant Multiple Rule

If $u$ is a differentiable function of $x$, and $c$ is a constant, then

$$\frac{d}{dx}(cu) = c\frac{du}{dx}.$$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \lim_{h\to 0}\frac{c-c}{h} = \lim_{h\to 0} 0 = 0.$$

$$\begin{aligned}\frac{d}{dx}cu &= \lim_{h\to 0}\frac{cu(x+h) - cu(x)}{h} \\ &= c\lim_{h\to 0}\frac{u(x+h) - u(x)}{h} \\ &= c\frac{du}{dx}\end{aligned}$$

### Derivative Sum Rule

If $u$ and $v$ are differentiable functions of $x$, then their sum $u + v$ is differentiable at every point where $u$ and $v$ are both differentiable. At such points,

$$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}.$$

### Derivative of the Natural Exponential Function

$$\frac{d}{dx}(e^x) = e^x$$

### Derivative Product Rule

If $u$ and $v$ are differentiable at $x$, then so is their product $uv$, anc

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + \frac{du}{dx}v.$$

### Derivative Quotient Rule

If $u$ and $v$ are differentiable at $x$ and if $v(x) \neq 0$, then the quotient $u/v$ is differentiable at $x$, and

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$

**EXAMPLE 1** Differentiate the following powers of $x$.

(a) $x^3$ (b) $x^{2/3}$ (c) $x^{\sqrt{2}}$ (d) $\dfrac{1}{x^4}$ (e) $x^{-4/3}$ (f) $\sqrt{x^{2+\pi}}$

**Solution**

(a) $\dfrac{d}{dx}(x^3) = 3x^{3-1} = 3x^2$

(b) $\dfrac{d}{dx}(x^{2/3}) = \dfrac{2}{3}x^{(2/3)-1} = \dfrac{2}{3}x^{-1/3}$

(c) $\dfrac{d}{dx}(x^{\sqrt{2}}) = \sqrt{2}\,x^{\sqrt{2}-1}$

(d) $\dfrac{d}{dx}\left(\dfrac{1}{x^4}\right) = \dfrac{d}{dx}(x^{-4}) = -4x^{-4-1} = -4x^{-5} = -\dfrac{4}{x^5}$

(e) $\dfrac{d}{dx}(x^{-4/3}) = -\dfrac{4}{3}x^{-(4/3)-1} = -\dfrac{4}{3}x^{-7/3}$

(f) $\dfrac{d}{dx}\left(\sqrt{x^{2+\pi}}\right) = \dfrac{d}{dx}\left(x^{1+(\pi/2)}\right) = \left(1+\dfrac{\pi}{2}\right)x^{1+(\pi/2)-1} = \dfrac{1}{2}(2+\pi)\sqrt{x^{\pi}}$

**Let $y = f(x)$ be a function of $x$. If the limit :**

$$\frac{dy}{dx} = f'(x) = \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} = \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}$$

**exists and is finite , we call this limit the derivative of $f$ at $x$ and say that $f$ is differentiable at $x$ .**

**EXAMPLE :** Find the derivative of the function : $f(x) = \dfrac{1}{\sqrt{2x+3}}$

**Sol.:**

$$f'(x) = \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} = \lim_{\Delta x\to 0}\frac{\dfrac{1}{\sqrt{2(x+\Delta x)+3}}-\dfrac{1}{\sqrt{2x+3}}}{\Delta x}$$

$$= \lim_{\Delta x\to 0}\frac{\sqrt{2x+3}-\sqrt{2(x+\Delta x)+3}}{\Delta x.\sqrt{2(x+\Delta x)+3}\sqrt{2x+3}}\cdot\frac{\sqrt{2x+3}+\sqrt{2(x+\Delta x)+3}}{\sqrt{2x+3}+\sqrt{2(x+\Delta x)+3}}$$

$$= \lim_{\Delta x\to 0}\frac{(2x+3)-(2(x+\Delta x)+3)}{\Delta x.\sqrt{2(x+\Delta x)+3}\sqrt{2x+3}(\sqrt{2x+3}+\sqrt{2(x+\Delta x)+3})}$$

$$= \frac{-2}{(2x+3)(\sqrt{2x+3}+\sqrt{2x+3})} = -\frac{1}{\sqrt{(2x+3)^3}}$$

**EXAMPLE** Find the derivative of **(a)** $y = \frac{t^2 - 1}{t^3 + 1}$, **(b)** $y = e^{-x}$

**Solution**

**(a)** We apply the Quotient Rule with $u = t^2 - 1$ and $v = t^3 + 1$:

$$\frac{dy}{dt} = \frac{(t^3 + 1) \cdot 2t - (t^2 - 1) \cdot 3t^2}{(t^3 + 1)^2} \qquad \frac{d}{dt}\left(\frac{u}{v}\right) = \frac{v(du/dt) - u(dv/dt)}{v^2}$$

$$= \frac{2t^4 + 2t - 3t^4 + 3t^2}{(t^3 + 1)^2}$$

$$= \frac{-t^4 + 3t^2 + 2t}{(t^3 + 1)^2}.$$

**(b)** $\frac{d}{dx}(e^{-x}) = \frac{d}{dx}\left(\frac{1}{e^x}\right) = \frac{e^x \cdot 0 - 1 \cdot e^x}{(e^x)^2} = \frac{-1}{e^x} = -e^{-x}$

**EXAMPLE** Find the derivative of the polynomial $y = x^3 + \frac{4}{3}x^2 - 5x + 1$

**Solution** $\frac{dy}{dx} = \frac{d}{dx}x^3 + \frac{d}{dx}\left(\frac{4}{3}x^2\right) - \frac{d}{dx}(5x) + \frac{d}{dx}(1)$

$= 3x^2 + \frac{4}{3} \cdot 2x - 5 + 0 = 3x^2 + \frac{8}{3}x - 5$

**EXAMPLE** Find the derivative of

$$y = \frac{(x - 1)(x^2 - 2x)}{x^4}$$

$$y = \frac{(x - 1)(x^2 - 2x)}{x^4} = \frac{x^3 - 3x^2 + 2x}{x^4} = x^{-1} - 3x^{-2} + 2x^{-3}$$

$$\frac{dy}{dx} = -x^{-2} - 3(-2)x^{-3} + 2(-3)x^{-4}$$

$$= -\frac{1}{x^2} + \frac{6}{x^3} - \frac{6}{x^4}.$$

**EXAMPLE** Find the derivative of **(a)** $y = \frac{1}{x}(x^2 + e^x)$, **(b)** $y = e^{2x}$.

**Solution**

**(a)** We apply the Product Rule with $u = 1/x$ and $v = x^2 + e^x$:

$$\frac{d}{dx}\left[\frac{1}{x}(x^2 + e^x)\right] = \frac{1}{x}(2x + e^x) + \left(-\frac{1}{x^2}\right)(x^2 + e^x)$$
$$= 2 + \frac{e^x}{x} - 1 - \frac{e^x}{x^2}$$
$$= 1 + (x - 1)\,\frac{e^x}{x^2}.$$

**(b)** $\frac{d}{dx}(e^{2x}) = \frac{d}{dx}(e^x \cdot e^x) = e^x \cdot \frac{d}{dx}(e^x) + \frac{d}{dx}(e^x) \cdot e^x = 2e^x \cdot e^x = 2e^{2x}$

**EXAMPLE :**

**EX-2- Find $\frac{dy}{dx}$ for the following functions :**

a) $y = (x^2 + 1)^5$

Sol.-

a) $\frac{dy}{dx} = 5(x^2 + 1)^4 . 2x = 10x(x^2 + 1)^4$

b) $y = [(5 - x)(4 - 2x)]^2$

Sol.

b) $\frac{dy}{dx} = 2[(5 - x)(4 - 2x)][-2(5 - x) - (4 - 2x)]$

$= 8(5 - x)(2 - x)(2x - 7)$

c) $y = (2x^3 - 3x^2 + 6x)^{-5}$

Sol.

c) $\frac{dy}{dx} = -5(2x^3 - 3x^2 + 6x)^{-6}(6x^2 - 6x + 6)$

$= -30(2x^3 - 3x^2 + 6x)^{-6}(x^2 - x + 1)$

$d)\ y=\frac{12}{x}-\frac{4}{x^3}+\frac{3}{x^4}$

**Sol.**

$d)\quad y=12x^{-1}-4x^{-3}+3x^{-4}\Rightarrow\frac{dy}{dx}=-12x^{-2}+12x^{-4}-12x^{-5}$

$$\Rightarrow\frac{dy}{dx}=-\frac{12}{x^2}+\frac{12}{x^4}-\frac{12}{x^5}$$

$e)\ y=\frac{(x^2+x)(x^2-x+1)}{x^3}$

**Sol.**

$e)\quad y=\frac{(x+1)(x^2-x+1)}{x^3}\Rightarrow$

$$\frac{dy}{dx}=\frac{x^3\left[(x^2-x+1)+(x+1)(2x-1)\right]-3x^2(x+1)(x^2-x+1)}{x^6}=-\frac{3}{x^4}$$

# Second- and Higher-Order Derivatives

### How to Read the Symbols for Derivatives

| | |
|---|---|
| $y'$ | "$y$ prime" |
| $y''$ | "$y$ double prime" |
| $\frac{d^2y}{dx^2}$ | "$d$ squared $y$ by $dx$ squared" |
| $y'''$ | "$y$ triple prime" |
| $y^{(n)}$ | "$y$ super $n$" |
| $\frac{d^ny}{dx^n}$ | "$d$ to the $n$ of $y$ by $dx$ to the $n$" |
| $D^n$ | "$d$ to the $n$" |

$$f''(x)=\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=y''=\frac{dy'}{dx}=\frac{d}{dx}(6x^5)=30x^4$$

**EXAMPLE**

The first four derivatives of $y = x^3 - 3x^2 + 2$ are

First derivative: $y' = 3x^2 - 6x$

Second derivative: $y'' = 6x - 6$

Third derivative: $y''' = 6$

Fourth derivative: $y^{(4)} = 0.$

## Home Works

**Derivative Calculations** find the first and second derivatives.

1. $y = -x^2 + 3$
2. $y = x^2 + x + 8$
3. $s = 5t^3 - 3t^5$
4. $w = 3z^7 - 7z^3 + 21z^2$
5. $y = \frac{4x^3}{3} - x + 2e^x$
6. $y = \frac{x^3}{3} + \frac{x^2}{2} + e^{-x}$
7. $w = 3z^{-2} - \frac{1}{z}$
8. $s = -2t^{-1} + \frac{4}{t^2}$

Find the derivatives of the functions in Exercises .

1. $y = \frac{2x + 5}{3x - 2}$
2. $g(x) = \frac{x^2 - 4}{x + 0.5}$
3. $v = (1 - t)(1 + t^2)^{-1}$
4. $f(s) = \frac{\sqrt{s} - 1}{\sqrt{s} + 1}$
5. $z = \frac{4 - 3x}{3x^2 + x}$
6. $f(t) = \frac{t^2 - 1}{t^2 + t - 2}$
7. $w = (2x - 7)^{-1}(x + 5)$
8. $u = \frac{5x + 1}{2\sqrt{x}}$

# Derivatives of Trigonometric Functions

## Derivatives of Trigonometric Functions

### Derivative of the Sine Function

$$\sin(x+h) = \sin x \cos h + \cos x \sin h.$$

If $f(x) = \sin x$, then

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} \\
&= \lim_{h\to 0}\frac{(\sin x\cos h + \cos x \sin h) - \sin x}{h} \\
&= \lim_{h\to 0}\frac{\sin x(\cos h - 1) + \cos x \sin h}{h} \\
&= \lim_{h\to 0}\left(\sin x \cdot \frac{\cos h - 1}{h}\right) + \lim_{h\to 0}\left(\cos x \cdot \frac{\sin h}{h}\right) \\
&= \sin x \cdot \underbrace{\lim_{h\to 0}\frac{\cos h - 1}{h}}_{\text{limit 0}} + \cos x \cdot \underbrace{\lim_{h\to 0}\frac{\sin h}{h}}_{\text{limit 1}} \\
&= \sin x \cdot 0 + \cos x \cdot 1 = \cos x.
\end{aligned}$$

## Derivatives of the Other Basic Trigonometric Functions

$$\tan x = \frac{\sin x}{\cos x}, \quad \cot x = \frac{\cos x}{\sin x}, \quad \sec x = \frac{1}{\cos x}, \quad \text{and} \quad \csc x = \frac{1}{\sin x}$$

### The derivatives of the other trigonometric functions:

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

**EXAMPLE** Find $y''$ if $y = \sec x$

$$y = \sec x$$

$$y' = \sec x \tan x$$

$$\begin{aligned} y'' &= \frac{d}{dx}(\sec x \tan x) \\ &= \sec x \frac{d}{dx}(\tan x) + \tan x \frac{d}{dx}(\sec x) \\ &= (\sec x)(\sec^2 x) + (\tan x)(\sec x \tan x) \\ &= \sec^3 x + \sec x \tan^2 x \end{aligned}$$

## Derivative of the Cosine Function

With the help of the angle sum formula for the cosine function

$$\cos(x + h) = \cos x \cos h - \sin x \sin h,$$

we can compute the limit of the difference quotient:

$$\begin{aligned}
\frac{d}{dx}(\cos x) &= \lim_{h \to 0} \frac{\cos(x + h) - \cos x}{h} \\
&= \lim_{h \to 0} \frac{(\cos x \cos h - \sin x \sin h) - \cos x}{h} \\
&= \lim_{h \to 0} \frac{\cos x(\cos h - 1) - \sin x \sin h}{h} \\
&= \lim_{h \to 0} \left(\cos x \cdot \frac{\cos h - 1}{h}\right) - \lim_{h \to 0} \left(\sin x \cdot \frac{\sin h}{h}\right) \\
&= \cos x \cdot \lim_{h \to 0} \frac{\cos h - 1}{h} - \sin x \cdot \lim_{h \to 0} \frac{\sin h}{h} \\
&= \cos x \cdot 0 - \sin x \cdot 1 \\
&= -\sin x.
\end{aligned}$$

## The derivative of the sine function is the cosine function:

$$\frac{d}{dx}(\sin x) = \cos x.$$

**EXAMPLE** We find derivatives of a difference, a product, and a quotient, each of which involves the sine function.

**(a)** $y = x^2 - \sin x$:

$$\frac{dy}{dx} = 2x - \frac{d}{dx}(\sin x) \qquad \text{Difference Rule}$$
$$= 2x - \cos x$$

**(b)** $y = e^x \sin x$:

$$\frac{dy}{dx} = e^x \frac{d}{dx}(\sin x) + \left(\frac{d}{dx}e^x\right)\sin x \qquad \text{Product Rule}$$
$$= e^x \cos x + e^x \sin x$$
$$= e^x(\cos x + \sin x)$$

**(c)** $y = \frac{\sin x}{x}$:

$$\frac{dy}{dx} = \frac{x \cdot \frac{d}{dx}(\sin x) - \sin x \cdot 1}{x^2} \qquad \text{Quotient Rule}$$
$$= \frac{x \cos x - \sin x}{x^2}$$

**EXAMPLE** We find derivatives of the cosine function in combinations with other functions.

**(a)** $y = 5e^x + \cos x$:

$$\frac{dy}{dx} = \frac{d}{dx}(5e^x) + \frac{d}{dx}(\cos x) \qquad \text{Sum Rule}$$
$$= 5e^x - \sin x$$

**(b)** $y = \sin x \cos x$:

$$\frac{dy}{dx} = \sin x \frac{d}{dx}(\cos x) + \left(\frac{d}{dx}(\sin x)\right)\cos x \qquad \text{Product Rule}$$
$$= (\sin x)(-\sin x) + (\cos x)(\cos x)$$
$$= \cos^2 x - \sin^2 x$$

**(c)** $y = \dfrac{\cos x}{1 - \sin x}$:

$$\frac{dy}{dx} = \frac{(1 - \sin x)\frac{d}{dx}(\cos x) - \cos x \frac{d}{dx}(1 - \sin x)}{(1 - \sin x)^2}$$
$$= \frac{(1 - \sin x)(-\sin x) - (\cos x)(0 - \cos x)}{(1 - \sin x)^2}$$
$$= \frac{1 - \sin x}{(1 - \sin x)^2} = \frac{1}{1 - \sin x}$$

$$\sin^2 x + \cos^2 x = 1$$

# HomeWorks:

1. $y = -10x + 3\cos x$

2. $y = \dfrac{3}{x} + 5\sin x$

3. $y = x^2 \cos x$

4. $y = \sqrt{x}\sec x + 3$

5. $y = -4\sqrt{x} + \dfrac{7}{e^x}$

6. $y = x^2 \cot x - \dfrac{1}{x^2}$

7. $f(x) = \sin x \tan x$

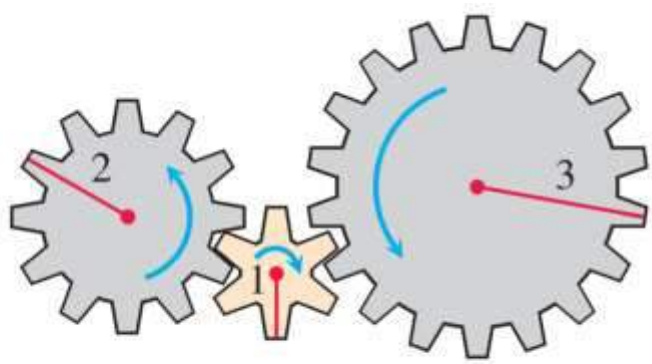
8. $g(x) = \dfrac{\cos x}{\sin^2 x}$

9. $y = xe^{-x}\sec x$

10. $y = (\sin x + \cos x)\sec x$

# Chain Rule

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$




C: $y$ turns B: $u$ turns A: $x$ turns

# Derivative of a Composite Function

**EXAMPLE** The function $y = (3x^2 + 1)^2$
is obtained by composing the functions $y = f(u) = u^2$ and $u = g(x) = 3x^2 + 1$
Calculating derivatives, we see that

$$\begin{aligned}\frac{dy}{du} \cdot \frac{du}{dx} &= 2u \cdot 6x \\ &= 2(3x^2 + 1) \cdot 6x \\ &= 36x^3 + 12x.\end{aligned}$$

**EXAMPLE** Show that the slope of every line tangent to the curve $y = 1/(1 - 2x)^3$ is positive.

**Solution** We find the derivative: Power Chain Rule with $u = (1 - 2x), n = -3$

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(1 - 2x)^{-3} \\ &= -3(1 - 2x)^{-4} \cdot \frac{d}{dx}(1 - 2x) \\ &= -3(1 - 2x)^{-4} \cdot (-2) \\ &= \frac{6}{(1 - 2x)^4}.\end{aligned}$$

**EXAMPLE 2** An object moves along the $x$-axis so that its position at any time $t \geq 0$ is given by $x(t) = \cos(t^2 + 1)$. Find the velocity of the object as a function of $t$.

**Solution** We know that the velocity is $dx/dt$. In this instance, $x$ is a composition of two functions: $x = \cos(u)$ and $u = t^2 + 1$. We have

$$\frac{dx}{du} = -\sin(u) \qquad x = \cos(u)$$

$$\frac{du}{dt} = 2t. \qquad u = t^2 + 1$$

By the Chain Rule,

$$\begin{aligned}\frac{dx}{dt} &= \frac{dx}{du} \cdot \frac{du}{dt} \\ &= -\sin(u) \cdot 2t \\ &= -\sin(t^2 + 1) \cdot 2t \\ &= -2t\sin(t^2 + 1).\end{aligned}$$

**EXAMPLE 6** The Power Chain Rule simplifies computing the derivative of a power of an expression.

**(a)** $\frac{d}{dx}(5x^3 - x^4)^7 = 7(5x^3 - x^4)^6 \frac{d}{dx}(5x^3 - x^4)$ Power Chain Rule with $u = 5x^3 - x^4$, $n = 7$

$= 7(5x^3 - x^4)^6(15x^2 - 4x^3)$

**(b)** $\frac{d}{dx}\left(\frac{1}{3x-2}\right) = \frac{d}{dx}(3x-2)^{-1}$

$= -1(3x-2)^{-2}\frac{d}{dx}(3x-2)$ Power Chain Rule with $u = 3x - 2$, $n = -1$

$= -1(3x-2)^{-2}(3)$

$= -\frac{3}{(3x-2)^2}$

In part (b) we could also find the derivative with the Quotient Rule.

**(c)** $\frac{d}{dx}(\sin^5 x) = 5\sin^4 x \cdot \frac{d}{dx}\sin x$ Power Chain Rule with $u = \sin x$, $n = 5$, because $\sin^n x$ means $(\sin x)^n$, $n \neq -1$

$= 5\sin^4 x \cos x$

**(d)** $\frac{d}{dx}\left(e^{\sqrt{3x+1}}\right) = e^{\sqrt{3x+1}} \cdot \frac{d}{dx}\left(\sqrt{3x+1}\right)$ Power Chain Rule with $u = 3x + 1$, $n = 1/2$

$= e^{\sqrt{3x+1}} \cdot \frac{1}{2}(3x+1)^{-1/2} \cdot 3$

$= \frac{3}{2\sqrt{3x+1}}e^{\sqrt{3x+1}}$

## Derivative Calculations

given $y = f(u)$ and $u = g(x)$,

1. $y = 6u - 9, \quad u = (1/2)x^4$
2. $y = 2u^3, \quad u = 8x - 1$
3. $y = \sin u, \quad u = 3x + 1$
4. $y = \cos u, \quad u = e^{-x}$
5. $y = \sqrt{u}, \quad u = \sin x$
6. $y = \sin u, \quad u = x - \cos x$
7. $y = \tan u, \quad u = \pi x^2$
8. $y = -\sec u, \quad u = \dfrac{1}{x} + 7x$

## Second Derivatives

Find $y''$ in Exercises

1. $y = \left(1 + \dfrac{1}{x}\right)^3$
2. $y = \left(1 - \sqrt{x}\right)^{-1}$
3. $y = \dfrac{1}{9}\cot(3x - 1)$
4. $y = 9\tan\left(\dfrac{x}{3}\right)$
5. $y = x(2x + 1)^4$
6. $y = x^2(x^3 - 1)^5$
7. $y = e^{x^2} + 5x$
8. $y = \sin(x^2 e^x)$

## Rules for Inequalities

If $a$, $b$, and $c$ are real numbers, then:

1. $a < b \Rightarrow a + c < b + c$
2. $a < b \Rightarrow a - c < b - c$
3. $a < b$ and $c > 0 \Rightarrow ac < bc$
4. $a < b$ and $c < 0 \Rightarrow bc < ac$
   Special case: $a < b \Rightarrow -b < -a$
5. $a > 0 \Rightarrow \frac{1}{a} > 0$
6. If $a$ and $b$ are both positive or both negative, then $a < b \Rightarrow \frac{1}{b} < \frac{1}{a}$

| | Notation | Set description | Type | Picture |
|---|---|---|---|---|
| **Finite:** | $(a, b)$ | $\{x \mid a < x < b\}$ | Open |  |
| | $[a, b]$ | $\{x \mid a \le x \le b\}$ | Closed |  |
| | $[a, b)$ | $\{x \mid a \le x < b\}$ | Half-open |  |
| | $(a, b]$ | $\{x \mid a < x \le b\}$ | Half-open |  |
| **Infinite:** | $(a, \infty)$ | $\{x \mid x > a\}$ | Open |  |
| | $[a, \infty)$ | $\{x \mid x \ge a\}$ | Closed |  |
| | $(-\infty, b)$ | $\{x \mid x < b\}$ | Open |  |
| | $(-\infty, b]$ | $\{x \mid x \le b\}$ | Closed |  |
| | $(-\infty, \infty)$ | $\mathbb{R}$ (set of all real numbers) | Both open and closed |  |

**EXAMPLE 1** Solve the following inequalities and show their solution sets on the real line.

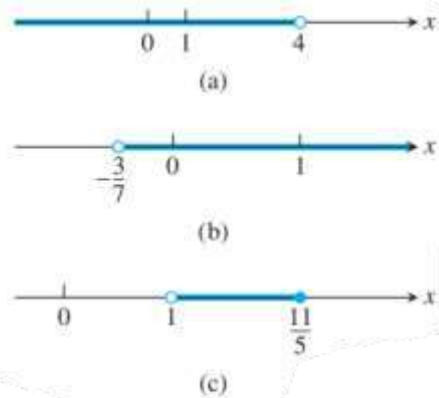
**(a)** $2x - 1 < x + 3$ **(b)** $-\frac{x}{3} < 2x + 1$ **(c)** $\frac{6}{x-1} \geq 5$

**Solution**

**(a)**
$$2x - 1 < x + 3$$
$$2x < x + 4$$
$$x < 4$$

The solution set is the open interval $(-\infty, 4)$ (Figure 1.1a).

**(b)**
$$-\frac{x}{3} < 2x + 1$$
$$-x < 6x + 3$$
$$0 < 7x + 3$$
$$-3 < 7x$$
$$-\frac{3}{7} < x$$




## Absolute Value

The **absolute value** of a number $x$, denoted by $|x|$, is defined by the formula

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0. \end{cases}$$

### Absolute Value Properties

1. $|-a| = |a|$
2. $|ab| = |a||b|$
3. $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$
4. $|a + b| \leq |a| + |b|$

### Absolute Values and Intervals

If $a$ is any positive number, then

5. $|x| = a$ if and only if $x = \pm a$
6. $|x| < a$ if and only if $-a < x < a$
7. $|x| > a$ if and only if $x > a$ or $x < -a$
8. $|x| \leq a$ if and only if $-a \leq x \leq a$
9. $|x| \geq a$ if and only if $x \geq a$ or $x \leq -a$

**EXAMPLE 2** Solving an Equation with Absolute Values

Solve the equation $|2x - 3| = 7$.

**Solution** By Property 5, $2x - 3 = \pm 7$, so there are two possibilities:

$$2x - 3 = 7 \qquad 2x - 3 = -7$$
$$2x = 10 \qquad 2x = -4$$
$$x = 5 \qquad x = -2$$

The solutions of $|2x - 3| = 7$ are $x = 5$ and $x = -2$.

**EXAMPLE 3** Solving an Inequality Involving Absolute Values

Solve the inequality $\left|5 - \frac{2}{x}\right| < 1$.

$$\left|5 - \frac{2}{x}\right| < 1 \Leftrightarrow -1 < 5 - \frac{2}{x} < 1$$
$$\Leftrightarrow -6 < -\frac{2}{x} < -4$$
$$\Leftrightarrow 3 > \frac{1}{x} > 2$$
$$\Leftrightarrow \frac{1}{3} < x < \frac{1}{2}.$$

**EXAMPLE** Solve the inequality and show the solution set on the real line:

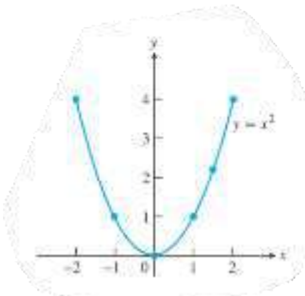
**(a)** $|2x - 3| \leq 1$ **(b)** $|2x - 3| \geq 1$

**(c)** $|y - 3| = 7$ **(d)** $|2t + 5| = 4$

# Functions

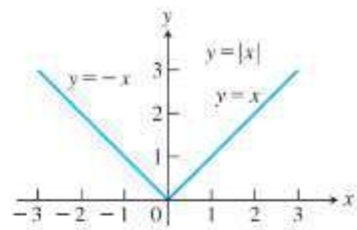
**EXAMPLE** Sketching a Graph

Graph the function $y = x^2$ over the interval $[-2, 2]$.




| $x$ | $y = x^2$ |
|---|---|
| $-2$ | 4 |
| $-1$ | 1 |
| 0 | 0 |
| 1 | 1 |
| $\frac{3}{2}$ | $\frac{9}{4}$ |
| 2 | 4 |

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$



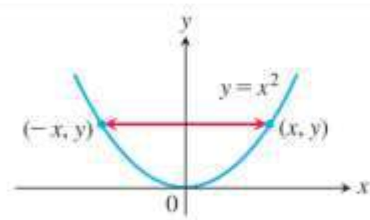

## Even Functions and Odd Functions:

**DEFINITIONS** A function $y = f(x)$ is an

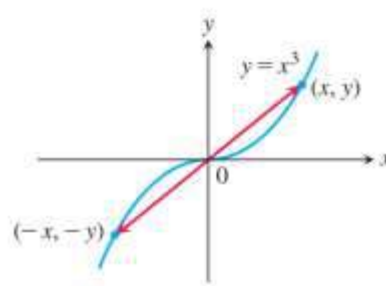
**even function of $x$** if $f(-x) = f(x)$,

**odd function of $x$** if $f(-x) = -f(x)$,

for every $x$ in the function's domain.




(a)



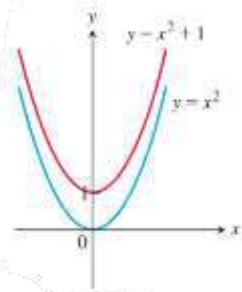

(b)

**EXAMPLE** Even function: $(-x)^2 = x^2$ for all $x$; symmetry about $y$-axis. So $f(-3) = 9 = f(3)$. Changing the sign of $x$ does not change the value of an even function.

Even function: $(-x)^2 + 1 = x^2 + 1$ for all $x$; symmetry about $y$-axis

$$f(x) = x^2$$

$$f(x) = x^2 + 1$$



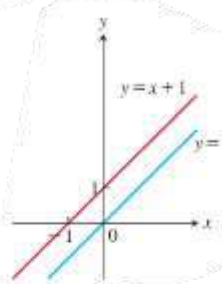

Odd function: $(-x) = -x$ for all $x$; symmetry about the origin. So $f(-3) = -3$ while $f(3) = 3$. Changing the sign of $x$ changes the sign of the value of an odd function.

Not odd: $f(-x) = -x + 1$, but $-f(x) = -x - 1$. The two are not equal.

Not even: $(-x) + 1 \neq x + 1$ for all $x \neq 0$

$$f(x) = x$$

$$f(x) = x + 1$$



# Homeworks:

**Defined Functions**

$$f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2 - x, & 1 < x \le 2 \end{cases}$$

$$g(x) = \begin{cases} 1 - x, & 0 \le x \le 1 \\ 2 - x, & 1 < x \le 2 \end{cases}$$

$$F(x) = \begin{cases} 4 - x^2, & x \le 1 \\ x^2 + 2x, & x > 1 \end{cases}$$

$$G(x) = \begin{cases} 1/x, & x < 0 \\ x, & 0 \le x \end{cases}$$

**Even and Odd Functions**

$$f(x) = 3$$

$$f(x) = x^2 + 1$$

$$g(x) = x^3 + x$$

$$f(x) = x^{-5}$$

$$f(x) = x^2 + x$$

$$g(x) = x^4 + 3x^2 - 1$$

$$g(x) = \frac{x}{x^2 - 1}$$

# LIMITS AND CONTINUITY

## Average and Instantaneous Speed

A moving body's **average speed** during an interval of time is found by dividing the distance covered by the time elapsed. The unit of measure is length per unit time: kilometers per hour, feet per second, or whatever is appropriate to the problem at hand.

$$\frac{\Delta y}{\Delta t} = \frac{16(t_0 + h)^2 - 16{t_0}^2}{h}.$$

**EXAMPLE 1** Finding an Average Speed

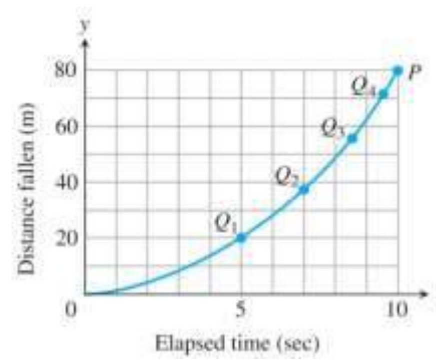
A rock breaks loose from the top of a tall cliff. What is its average speed

**(a)** during the first 2 sec of fall?

**(b)** during the 1-sec interval between second 1 and second 2?

**(a)** For the first 2 sec: $\frac{\Delta y}{\Delta t} = \frac{16(2)^2 - 16(0)^2}{2 - 0} = 32 \frac{\text{ft}}{\text{sec}}$

**(b)** From sec 1 to sec 2: $\frac{\Delta y}{\Delta t} = \frac{16(2)^2 - 16(1)^2}{2 - 1} = 48 \frac{\text{ft}}{\text{sec}}$

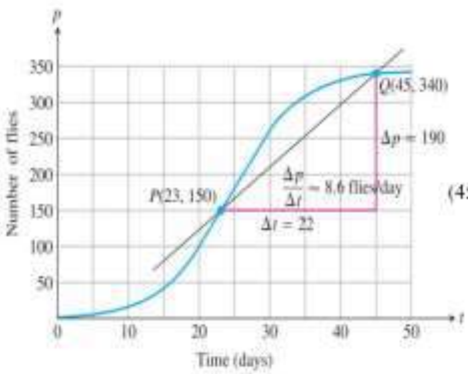


# Average Rates of Change and Secant Lines

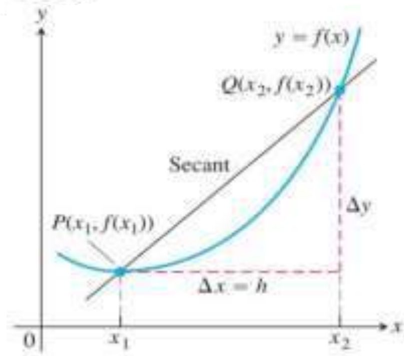
### Average Rate of Change over an Interval

The **average rate of change** of $y = f(x)$ with respect to $x$ over the interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h}, \qquad h \neq 0.$$




(45, 340) $\frac{340 - 150}{45 - 23} \approx 8.6$




**THEOREM 1—Limit Laws**
If $L$, $M$, $c$, and $k$ are real numbers and

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = M, \quad \text{then}$$

1. *Sum Rule:* $\lim_{x \to c} (f(x) + g(x)) = L + M$
2. *Difference Rule:* $\lim_{x \to c} (f(x) - g(x)) = L - M$
3. *Constant Multiple Rule:* $\lim_{x \to c} (k \cdot f(x)) = k \cdot L$
4. *Product Rule:* $\lim_{x \to c} (f(x) \cdot g(x)) = L \cdot M$
5. *Quotient Rule:* $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}, \quad M \neq 0$
6. *Power Rule:* $\lim_{x \to c} [f(x)]^n = L^n$, $n$ a positive integer
7. *Root Rule:* $\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{1/n}$, $n$ a positive integer

(If $n$ is even, we assume that $f(x) \geq 0$ for $x$ in an interval containing $c$.)

7). $\lim_{x \to \infty} \frac{1}{x} = 0$

8). $\lim_{x \to 0^+} \frac{1}{x} = \infty$

9). $\lim_{x \to 0^-} \frac{1}{x} = -\infty$

10). $\lim_{x \to 0} \sin x = 0$

11). $\lim_{x \to 0} \cos x = 1$

12). $\lim_{x \to 0} \frac{\sin x}{x} = 1$ $\Rightarrow \lim_{x \to 0} \frac{x}{\sin x} = 1$

13). $\lim_{x \to \infty} \frac{\cos x}{x} = 0$

14). $\lim_{x \to \infty} \frac{\sin x}{x} = 0$

15). Undefined Expression مقادير غير معرفة

( $\frac{0}{0}$ , $0^0$ , $\infty^0$ , $\frac{\infty}{\infty}$ , $\infty - \infty$ , $0 \times \infty$ ,

but we can say $\infty + \infty = \infty$ ).

**EXAMPLE** Use the observations $\lim_{x \to c} k = k$ and $\lim_{x \to c} x = c$

**(a)** $\lim_{x \to c} (x^3 + 4x^2 - 3)$ **(b)** $\lim_{x \to c} \frac{x^4 + x^2 - 1}{x^2 + 5}$ **(c)** $\lim_{x \to -2} \sqrt{4x^2 + 3}$

**Solution**

**(a)**
$$\lim_{x \to c} (x^3 + 4x^2 - 3) = \lim_{x \to c} x^3 + \lim_{x \to c} 4x^2 - \lim_{x \to c} 3 = c^3 + 4c^2 - 3$$

**(b)**
$$\lim_{x \to c} \frac{x^4 + x^2 - 1}{x^2 + 5} = \frac{\lim_{x \to c} (x^4 + x^2 - 1)}{\lim_{x \to c} (x^2 + 5)} = \frac{\lim_{x \to c} x^4 + \lim_{x \to c} x^2 - \lim_{x \to c} 1}{\lim_{x \to c} x^2 + \lim_{x \to c} 5} = \frac{c^4 + c^2 - 1}{c^2 + 5}$$

**(c)**
$$\lim_{x \to -2} \sqrt{4x^2 + 3} = \sqrt{\lim_{x \to -2} (4x^2 + 3)} = \sqrt{\lim_{x \to -2} 4x^2 + \lim_{x \to -2} 3} = \sqrt{4(-2)^2 + 3} = \sqrt{16 + 3} = \sqrt{19}$$

Example : Evaluate limits of the following

(d) $\lim_{x\to 9}\frac{\sqrt{x}-3}{x-9}=\frac{\sqrt{9}-3}{9-9}=\frac{0}{0}$

$$\lim_{x\to 9}\frac{\sqrt{x}-3}{(\sqrt{x}-3)(\sqrt{x}+3)}$$

$$=\lim_{x\to 9}\frac{1}{(\sqrt{x}+3)}=\frac{1}{(\sqrt{9}+3)}$$

$$=\frac{1}{3+3}=\frac{1}{6}$$

(e) $\lim_{x\to 2}\frac{4-x^2}{3-\sqrt{x^2+5}}=\frac{4-4}{3-\sqrt{4+5}}=\frac{0}{0}$

$$\lim_{x\to 2}\frac{4-x^2}{3-\sqrt{x^2+5}}\cdot\frac{(3+\sqrt{x^2+5})}{(3+\sqrt{x^2+5})}$$

$$=\lim_{x\to 2}\frac{(4-x^2)(3+\sqrt{x^2+5})}{9-(x^2+5)}$$

$$=\lim_{x\to 2}\frac{(4-x^2)(3+\sqrt{x^2+5})}{(4-x^2)}$$

$$=\lim_{x\to 2}(3+\sqrt{x^2+5})=(3+\sqrt{4+5})=3+\sqrt{9}$$

$$=6$$

**(f)** $\lim_{x\to 0}\frac{5x^3+8x^2}{3x^4-16x^2}=\frac{5(0)^3+8(0)^2}{3(0)^4-16(0)^2}=\frac{0}{0}$

$$\lim_{x\to 0}\frac{x^2(5x+8)}{x^2(3x^2-16)}$$

$$=\lim_{x\to 0}\frac{5x+8}{3x^2-16}$$

$$=\frac{5(0)+8}{3(0)^2-16}$$

$$=\frac{8}{-16}=-\frac{1}{2}$$

**(g)** $\lim_{x\to 1}\frac{x-1}{\sqrt{x^2+3}-2}=\frac{1-1}{\sqrt{1+3}-2}=\frac{0}{\sqrt{4}-2}=\frac{0}{0}$

$$\lim_{x\to 1}\frac{(x-1)}{\sqrt{x^2+3}-2}\cdot\frac{\sqrt{x^2+3}+2}{\sqrt{x^2+3}+2}=\lim_{x\to 1}\frac{(x-1)(\sqrt{x^2+3}+2)}{(x^2+3)-4}$$

$$=\lim_{x\to 1}\frac{(x-1)(\sqrt{x^2+3}+2)}{(x^2-1)}=\lim_{x\to 1}\frac{(x-1)(\sqrt{x^2+3}+2)}{(x-1)(x+1)}$$

$$=\lim_{x\to 1}\frac{\sqrt{x^2+3}+2}{(x+1)}=\frac{\sqrt{1+3}+2}{1+1}=\frac{\sqrt{4}+2}{2}=\frac{4}{2}=2$$

# Limits at Infinity of Rational Function

لحل هذا النوع من الغايات سوف نتطرق الى ثلاث حالات وهي كالاتي:

1. إذا كانت درجة الاس في البسط والمقام متساوية سوف نقسم على اعلى اس في الدالة والناتج يكون عدد حقيقي.
2. إذا كانت درجة الاس في البسط اقل من المقام فسوف نقسم الدالة على أكبر اس موجود والناتج يكون مساوي الى صفر.
3. إذا كانت درجة الاس في البسط أكبر من المقام نقسم الدالة على أكبر اس موجود فيها والناتج يقترب الى ∞ .

Example : find $\lim\limits_{x\to\infty} \frac{x^2-2}{2x^3-3x^2-5}$

Sol :

$$\lim_{x\to\infty} \frac{x^2-2}{2x^3-3x^2-5} = \frac{\infty^2-2}{2(\infty)^3-3(\infty)^2-5} = \frac{\infty}{\infty}$$

$$\lim_{x\to\infty} \frac{\frac{x^2}{x^3}-\frac{2}{x^3}}{\frac{2x^3}{x^3}-\frac{3x^2}{x^3}-\frac{5}{x^3}} = \lim_{x\to\infty} \frac{\frac{1}{x}-\frac{2}{x^3}}{2-\frac{3}{x}-\frac{5}{x^3}} = \frac{\frac{1}{\infty}-\frac{2}{\infty^3}}{2-\frac{3}{\infty}-\frac{5}{\infty^3}} = \frac{0}{2} = 0$$

Example : find $\lim_{x\to\infty} \frac{2x+3}{5x+7}$

Sol :

$$\lim_{x\to\infty} \frac{2x+3}{5x+7} = \frac{2(\infty)+3}{5(\infty)+7} = \frac{\infty}{\infty}$$

$$\lim_{x\to\infty} \frac{\frac{2x}{x}+\frac{3}{x}}{\frac{5x}{x}+\frac{7}{x}} = \lim_{x\to\infty} \frac{2+\frac{3}{x}}{5+\frac{7}{x}} = \frac{2+\frac{3}{\infty}}{5+\frac{7}{\infty}} = \frac{2+0}{5+0} = \frac{2}{5}$$

# Inequalities

# Inequalities

An inequality is like an equation, but instead of an equal sign (=) it has one of these signs:

- $<$ : less than
- $\leq$ : less than or equal to
- $>$ : greater than
- $\geq$ : greater than or equal to

# Graphing Inequalities

- When we graph an inequality on a number line we use open and closed circles to represent the number.

$<$ $>$ Plot an open circle

$\leq$ $\geq$ Plot a closed circle

## Graphing Rules

| Symbol | Circle | Direction of Arrow |
|---|---|---|
| < | Open | Left |
| > | Open | Right |
| ≤ | Closed | Left |
| ≥ | Closed | Right |

## "x < 5"

means that whatever value *x* has,

it must be less than 5.

What could x be?

# “x ≥ -2”

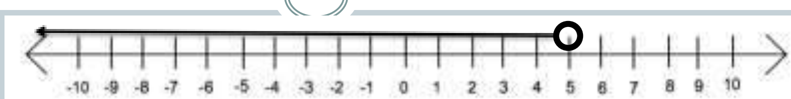
means that whatever value *x* has,

it must be greater than <u>or</u> equal to -2.

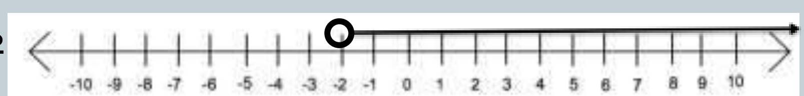
What could x be?

# Examples:

- x < 5

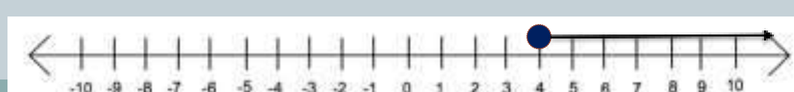



- x > -2



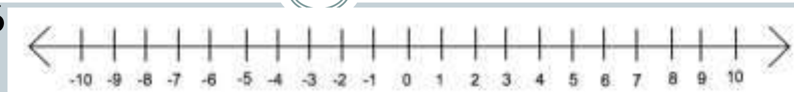

- x ≤ -8


- x ≥ 4

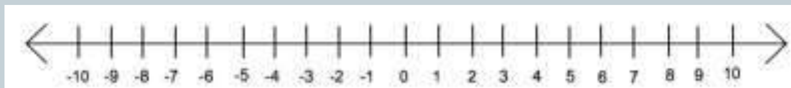


# You Try:

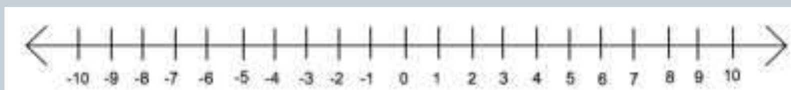
- x < -6




- x > 2




- x ≤ 0




- x ≥ -7

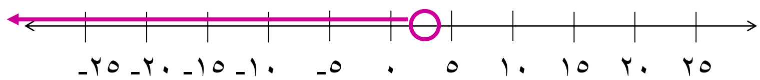
# Solve an Inequality

w + ٥ < ٨

\- ٥ -٥

w < ٣

All numbers less than ٣ are solutions to this problem!



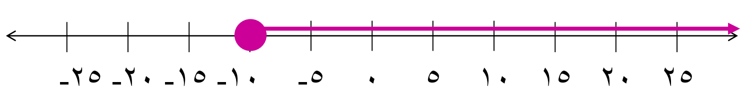

# More Examples

٨ + r ≥ -٢

-٨ -٨

r ≥ -١٠

ll numbers greater than-١٠ (including -١٠)



# More Examples

٢x > -٢

٢ ٢

x > -١

All numbers greater than -١ make this problem true!


# More Examples

٢h + ٨ ≤ ٢٤

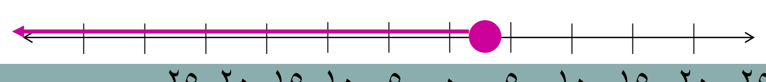
-٨ -٨

٢h ≤ ١٦

٢ ٢

h ≤ ٨

All numbers less than ٨ (including ٨)



# Solving One-Step Inequalities

X-١٥<٧٣

# Solving One-Step Inequalities

$y+١٥<٢٥$

# Solving One-Step Inequalities

$$\frac{X}{٥} = ١٠$$

Multiply both sides by the reciprocal of the coefficient

$$٥ \cdot \frac{X}{٥} = ١٠ \cdot ٥$$

$$X = ٥٠$$

# Solving One-Step Inequalities

٥x □ ٢٠

Divide both sides by the coefficient of x

$$\frac{٥x}{٥} \;□\; \frac{٢٠}{٥}$$

x □ ٤

## Solving Inequalities!

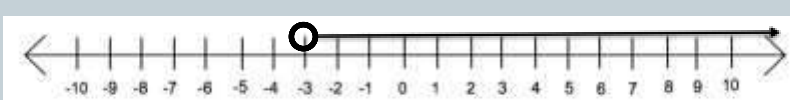
- Solving inequalities is the same as solving equations.
- There are only 2 things you need to know…
  - 1.) If you multiply or divide by a negative number you must switch the sign.

$$\frac{-7x}{-7} < \frac{21}{-7}$$

$$x > -3$$

Dividing by a negative means switch the sign!!

  - 2.) You will graph your solutions.



## Special Case 1: Switching the Signs

- When solving inequalities, if you **multiply or divide by a negative** you must switch the signs.
- Switching the signs:
  - Less than becomes Greater than < **switches to** >
  - Greater than becomes Less than > **switches to** <
  - Less than or equal to becomes Greater than or equal to ≤ **switches to** ≥
  - Greater than or equal to becomes Less than or equal to ≥ **switches to** ≤

## Division Property for Inequalities

Caution! Dividing by a negative number

$-٥x$ □ ٢٠

Same if multiplying?

$\frac{-٥x}{-٥}$ □ $\frac{٢٠}{-٥}$ → $x$ □ $-٤$

Notice: Sign CHANGED

## Multiplication Property for Inequalities

**YES!**

Caution! When you multiply by a negative number…

$\frac{-x}{5} > 2$

…the sign CHANGES

$\left(\frac{-5}{1}\right)\frac{-x}{5} > 2(-5)$ → $x < -10$

## Solving One-Step Inequalities

Let's try some on our own …… ready?

## Solving One-Step Inequalities #1

$X$ + ٦ □ ٧

## Solving One-Step Inequalities #2

٣ □ $X$ − ٥

## Solving One-Step Inequalities #3

−٣X □ −١٥

## Solving One-Step Inequalities #4

X − ٩ □ −٥

# Answers for #1 - #4

١. x ≤ ١

٢. ٨ ≤ x or x ≥ ٨

٣. x ≤ ٥

٤. x > ٤

# Solving One-Step Inequalities #5

$$\frac{١}{٢}x \square -٣$$

## Solving One-Step Inequalities #6

$$-٣ \square \frac{x}{-٧}$$

## Solving One-Step Inequalities #7

$$٥ + x \square ٧$$

# Solving One-Step Inequalities #8

$$\frac{x}{٣} \;\square\; ٥$$

# Answers for #5 - #8

٥. x ≥ -٦

٦. ٢١ < x or x > ٢١

٧. x ≥ ٢

٨. x > ١٥

# Solving Inequalities

- Follow same steps used to solve equations:

$$3x + 4 < 13$$
$$-4 \quad -4$$
$$\frac{3x}{3} < \frac{9}{3}$$
$$x < 3$$

# Practice

- $-2x + 5 \geq 15$
- $14 > \frac{x}{-4} + 4$
- $17 - 3x < 41$
- $\frac{x}{-5} \leq 7$

## Time to Practice!

- Solve:

$$6x - 8 > 22$$

## Practice Problem 1

$$-4 - 5v < -29$$

## Practice Problem 2

$$-1 + 4x \leq 31$$

## Practice Problem 3

$$\frac{-2 + r}{9} > -1$$

## Practice Problem 4

$$-52 < 8 - 5k$$

## Practice Problem 5

$$8 - 7n > -20$$

## Practice Problem 6

$$-9 \geq -8 + v$$
$$\underline{\quad -6}$$

## Home Work

- Solve the inequality and graph the answer.

1. x + 3 > -4
2. 6d ≥ 24
3. 2x - 8 < 14
4. -2c – 4 ≤ 2

١. x > -٧
٢. d ≥ ٤
٣. x < ١١
٤. c ≤ -٣

# Practice

- $x + 5 \geq 13$
- $5 + x < 7$
- $2x - 14 > 4$
- $x/4 + 3 \leq 7$